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Separation of variables for the $\mathcal{D}_n$-type periodic Toda lattice

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Abstract. We prove separation of variables for the most general ($\mathcal{D}_n$-type) periodic Toda lattice with a 2×2 Lax matrix. It is achieved by finding proper normalization for the corresponding Baker–Akhiezer function. Separation of variables for all other periodic Toda lattices associated with infinite series of root systems follows by taking appropriate limits.

1. Introduction

Bogoyavlensky [3] introduced periodic Toda lattices corresponding to the root systems of affine algebras. In this case the integrable potentials in the Hamiltonian

$$H = \sum_{j=1}^{n} \frac{p_j^2}{2} + V(q) \qquad \{p_j, q_k\} = \delta_{jk} \tag{1.1}$$

for the loop algebras $\mathcal{A}_n^{(1)}$, $\mathcal{B}_n^{(1)}$, $\mathcal{C}_n^{(1)}$, and $\mathcal{D}_n^{(1)}$ have the form

$$V_{\mathcal{A}_n^{(1)}} = V_{\mathcal{A}_n} + \exp(q_n - q_1)$$

$$V_{\mathcal{B}_n^{(1)}} = V_{\mathcal{A}_n} + \exp(q_n) + \exp(-q_1 - q_2)$$

$$V_{\mathcal{C}_n^{(1)}} = V_{\mathcal{A}_n} + \exp(2q_n) + \exp(-2q_1)$$

$$V_{\mathcal{D}_n^{(1)}} = V_{\mathcal{A}_n} + \exp(q_{n-1} + q_n) + \exp(-q_1 - q_2)$$

where

$$V_{\mathcal{A}_n} = \sum_{j=1}^{n-1} \exp(q_j - q_{j+1})\,.$$

For the twisted loop algebras the integrable potentials are as follows [21]:

$$V_{\mathcal{A}_{2n}^{(2)}} = V_{\mathcal{A}_n} + \exp(q_n) + \exp(-2q_1)$$

$$V_{\mathcal{A}_{2n+1}^{(2)}} = V_{\mathcal{A}_n} + \exp(-q_1 - q_2) + \exp(2q_n)$$

$$V_{\mathcal{D}_{n+1}^{(2)}} = V_{\mathcal{A}_n} + \exp(q_n) + \exp(-q_1)\,.$$

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Inozemtsev [7] found a generic ($\mathcal{D}_n$-type) periodic Toda lattice with four more parameters ($A,\ B,\ C,\ D$) in the potential

$$V(q) = V_{\mathcal{A}_n} + \exp(-q_1 - q_2) + \exp(q_{n-1} + q_n) + \frac{A}{\sinh^2(q_1/2)} + \frac{B}{\sinh^2 q_1} + \frac{C}{\sinh^2(q_n/2)} + \frac{D}{\sinh^2 q_n} \tag{1.2}$$

which includes all the above potentials as limiting cases. He gave the $2n \times 2n$ Lax representation and proved Liouville integrability for this system.

Sklyanin [23] found 2×2 Lax representations for all cases (including $\mathcal{BC}_n$) except $\mathcal{B}_n^{(1)}$, $\mathcal{D}_n^{(1)}$, $\mathcal{A}_{2n+1}^{(2)}$, and (1.2), introducing a reflection equation which also provided quantization of those systems. The 2×2 Lax matrices (L-operators) for the remaining three cases and for Inozemtsev's extension (1.2) were found in [10, 11, 20]. See also [14] where Inozemtsev's case was interpreted as the $\mathcal{A}_n$-type open Toda lattice interacting with two Lagrange tops (one on each end of the lattice).

The periodic Toda lattice (of $\mathcal{A}_n^{(1)}$ type) was separated in [6]. In [22] it was treated within the R-matrix method which allowed separation of its quantum counterpart. Partial results on the separation of variables for other Toda lattices are scattered in several places [11, 20, 14], essentially repeating the basic technique of [22] for the case of reflection equation algebra introduced in [23]. For a detailed algebro-geometrical treatment of many of these Toda lattices we refer the reader to [1].

In the present paper we prove separation of variables for the generic potential (1.2) with the 2×2 Lax matrix $L(u)$. It is achieved by finding proper normalization for the corresponding Baker–Akhiezer (BA) function $f(u)$

$$L(u)\, f(u) = v\, f(u) \qquad \big(f(u) = (f_1(u),\, f_2(u))^t\big)\,.$$

We recall that (usually) the separation variables are obtained as poles of the BA function (cf the review [25]). The standard normalization $f_1(u) = 1$ (or $f_2(u) = 1$) which was valid, for instance, for the $\mathcal{A}_n^{(1)}$ case [22] does not work here, giving too many poles which are not in involution with respect to the Poisson bracket. The reason is extra symmetries of the Lax matrix. To obey symmetry and reduce the number of poles to the number of degrees of freedom, one has to find a specific normalization $\boldsymbol{\alpha}(u) = (\alpha_1(u), \alpha_2(u))$ of the BA vector:

$$\alpha_1(u)\, f_1(u) + \alpha_2(u)\, f_2(u) = 1\,.$$

The structure of the paper is as follows. In section 2 we give an overview of the method of separation of variables, applying it in section 3 to the integrable system in question. In section 4 there are some concluding remarks.

2. The method

The method of separation of variables plays an important role in studying Liouville integrable systems.

Definition 1. A Liouville integrable system possesses a Lax matrix if there is a matrix $L(u)$ dependent on a 'spectral parameter' $u \in \mathbb{C}$ such that its characteristic polynomial obeys two conditions:
(i) Poisson involutivity: $\{\det(L(u) - v\cdot \mathbb{1}), \det(L(\tilde u) - \tilde v \cdot \mathbb{1})\} = 0, \quad \forall\, u, \tilde u, v, \tilde v \in \mathbb{C}$;
(ii) $\det(L(u) - v\cdot\mathbb{1})$ generates all integrals of motion H_i.

Definition 2. By *separation of variables* (SoV) in classical mechanics we mean the existence of a canonical transformation $M : (x, p) \mapsto (u, v)$, $M : H_i(x, p) \mapsto H_i(u, v)$ such that $H_i(u, v)$ are in the separated form

$$\Phi(u_i, v_i; H_1, \ldots, H_n) \equiv \det(L(u_i) - v_i \cdot \mathbb{1}) = 0 \qquad i = 1, \ldots, n\,.$$

The above definition corresponds precisely to the standard definition of SoV in the Hamilton–Jacobi equation [2].

We would like to note here that we have connected our definition of SoV with the Lax representation and with the associated spectral curve of the Lax matrix $L(u)$, so it might be not unique (if it exists) in the case where a chosen integrable system has, for instance, two or more inequivalent Lax representations.

One of the main questions in the theory is: how does one constructively define those new *separation variables* (u_j, v_j) sitting on the spectral curve of an L-matrix for a given integrable system?

For a very long time a great deal of attention has been given to so-called coordinate separation of variables or to separation in the configuration space (see, for instance, [8, 24, 12, 13, 4, 25] and references therein). In this case the separation variables u_j are functions of the x_i alone:

$$u_j = u_j(x_1, \ldots, x_n)\,. \tag{2.1}$$

Such kinds of integrable systems admitting a coordinate (spatial) separation of variables have been studied in detail, although at the same time it was understood that by no means every Liouville integrable system can be separated through a transition (2.1) to new 'coordinates' u_i. The class of admissable transformations should be enlarged for a generic integrable system up to a general canonical transformation

$$u_j = u_j(x_1, \ldots, x_n, p_1, \ldots, p_n) \qquad v_j = v_j(x_1, \ldots, x_n, p_1, \ldots, p_n)\,. \tag{2.2}$$

The very existence of SoV according to the above definition is still unproved in general, to author's knowledge; although there are powerful methods which have been applied to many families of integrable systems (see the recent review [25]) showing that separability is one of the most important features of integrability, and that hopefully the latter always implies the former. The method of SoV in its modern formulation can be found in [25]; see also [16–19, 15]. Here we describe its main steps very briefly.

The first difficulty is: *how does one find the separation variables u_j?* There is a general answer to this question, which has been inspired by the whole experience of the inverse scattering method, and it is a very simple one:

Answer. The u_j are poles of the Baker–Akhiezer function which is properly normalized.

There is, however, a slight further problem of choosing the correct normalization for the BA function; the problem which was not completely solved by powerful and successful method of inverse scattering. So, a general theory connecting the symmetry of the Lax matrix to proper normalization vector of the BA function is still incomplete. However, supposing that one somehow knows the correct normalization, then one could proceed further and put the above general recipe into the formulae (cf [25]).

The linear problem for the BA function $f(u)$ is of the form

$$L(u)\, f(u) = v(u)\, f(u) \qquad (\det(L(u) - v \cdot \mathbb{1}) = 0)\,. \tag{2.3}$$

The normalization $\boldsymbol{\alpha}(u)$ of the eigenvectors $f(u)$ has to be fixed

$$\sum_{i=1}^{N} \alpha_i(u)\, f_i(u) = 1 \qquad (f(u) \equiv (f_1(u), \ldots, f_N(u))^t)\,. \tag{2.4}$$

Let $L(u)$ be a meromorphic function in u then $f(u)$ is also meromorphic in u. Let us look at its ($f(u)$'s) poles u_j:

$$f_i^{(j)} = \operatorname*{res}_{u=u_j} f_i(u)\,.$$

Then from equations (2.3), (2.4) we have

$$\begin{aligned} &L(u_j)\, f^{(j)} = v_j\, f^{(j)} \qquad v_j \equiv v(u_j) \\ &\sum_{i=1}^{N} \alpha_i(u_j)\, f_i^{(j)} = 0\,. \end{aligned} \tag{2.5}$$

Equations (2.5) are $N+1$ linear homogeneous equations for the separation variables $u = u_j$ and $v = v_j$ which are bounded by definition to the spectral curve (cf equation (2.3)). These equations have to be compartible. The compatibility of the system (2.5) is equivalent to the condition

$$\operatorname{rank}\begin{pmatrix} \boldsymbol{\alpha}(u) \\ L(u) - v\cdot \mathbb{1} \end{pmatrix} = N-1 \tag{2.6}$$

where $\boldsymbol{\alpha}$ is thought of as a row vector. Finally, the condition (2.6) can be rewritten as the following vector equation:

$$\boldsymbol{\alpha}\cdot (L(u) - v\cdot \mathbb{1})^{\wedge} = 0 \tag{2.7}$$

where the wedge denotes the classical adjoint matrix (matrix of cofactors).

Proposition 1. Excluding v, one can derive from equations (2.7) the equation for u in the form

$$B(u) = \det\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha}\cdot L(u) \\ \vdots \\ \boldsymbol{\alpha}\cdot L^{N-1}(u) \end{pmatrix} = 0\,. \tag{2.8}$$

Proof. When $u = u_j$ we have the equations (cf equations (2.5))

$$L(u)\, f = v(u)\, f \qquad \boldsymbol{\alpha}\, f = 0\,. \tag{2.9}$$

Hence

$$\boldsymbol{\alpha}\, L^k\, f = 0 \qquad k = 0, 1, 2, \ldots\,.$$

Then equation (2.8) follows because f is a non-zero vector. □

Also, from equations (2.7) we can obtain formulae for v in the form

$$v = A(u)$$

where $A(u)$ are some rational functions of the entries of $L(u)$ (cf [15]).

What is left is just to verify (somehow) the canonical brackets between the whole set of separation variables, namely between the zeros u_j of $B(u)$ and their conjugate variables $v_j \equiv v(u_j) = A(u_j)$. To do this final calculation we need information about the Poisson brackets between entries of the Lax matrix $L(u)$, which is usually provided by corresponding r-matrix (standard or dynamical).

In order to perform a SoV, say, in a strong sense, one also has to try to obtain an explicit representation for the corresponding generating function $F(u|x)$ of the separating canonical transform M from the set (x_j, p_j) to the set (u_j, v_j). Actually, to find the generating function $F(u|x)$ one has to solve the system of non-linear equations of the form

$$\boldsymbol{\alpha}(u_j)\left(L(u_j) + \frac{\partial F}{\partial u_j}\cdot \mathbb{1}\right)^{\wedge}\Bigg|_{p_k=\partial F/\partial x_k} = 0 \qquad j = 1, \ldots, n\,.$$

In the quantum case the function $F(u|x)$ has a quantum counterpart: the kernel $\mathcal{M}_{\hbar}(u|x)$ of the separating integral transform $M_{\hbar}$, so that

$$\mathcal{M}_{\hbar}(u|x) \sim \exp\left(\frac{i}{\hbar}F(u|x)\right) \qquad \hbar \to 0\,.$$

For some integrable systems such special functions of many variables (F and $\mathcal{M}_{\hbar}$) can be obtained in very explicit terms (cf [25, 16, 17, 18]). We remark here that the other generating function has traditionally been associated with constructions of the method of separation of variables in the Hamilton–Jacobi equation, namely the action function $S(H|u)$ given in terms of separation variables, u_j, and integrals of motion, H_j. Our choice of arguments of the generating function is justified by the quantum case where $F(y|x)$ has a direct quantum analogue, while the action function $S(H|u)$ does not have such a nice quantum counterpart at all.

Very often the above prescription of SoV should be read 'in the opposite direction' (because one does not usually know the separating normalization in advance). Sometimes, regardless of the choice of the vector $\boldsymbol{\alpha}$, the Baker–Akhiezer function $f(u)$ has exactly the required number of poles in involution. Sometimes, and this is very important, $f(u)$ has too many poles and they are *not* mutually in involution, showing that there are some constraints between them. In the latter case, one should find a proper (and quite unique) normalization vector $\boldsymbol{\alpha}(u)$ so that all the extra poles of $f(u)$ are fixed as *constants*. The prescription then makes us search for a way to resolve possible constraints on poles of the Baker–Akhiezer function by using the freedom to choose its normalization. In this paper we show that this is the case in the $\mathcal{D}_n$-type periodic Toda lattice and give the correct normalization for corresponding $f(u)$, thereby producing a SoV for this system, which has not been solved before by this method.

If we make a similarity transformation for the L-matrix

$$\widetilde{L}(u) = V(u)\, L(u)\, V^{-1}(u)$$

with a non-degenerate matrix $V(u)$ then the linear problem

$$L(u)\, f(u) = v(u)\, f(u) \qquad \boldsymbol{\alpha}\cdot f = 1 \tag{2.10}$$

turns into

$$\widetilde{L}(u)\, \widetilde{f}(u) = v(u)\, \widetilde{f}(u) \qquad \boldsymbol{\alpha}_0 \cdot \widetilde{f} = 1$$

where

$$\widetilde{f}(u) = V(u)\, f(u) \qquad \boldsymbol{\alpha}(u) = \boldsymbol{\alpha}_0(u)\, V(u)\,. \tag{2.11}$$

This shows that the freedom to choose the normalization vector $\boldsymbol{\alpha}$ is equivalent to the freedom to make similarity transformations to the initial Lax representation.

Let us put $N = 2$, so that we assume from now on that we have a 2×2 Lax representation for our integrable system. In this case the equations of SoV (2.6) have the form

$$\operatorname{rank}\begin{pmatrix} \alpha_1(u) & \alpha_2(u) \\ L_{11}(u) - v & L_{12}(u) \\ L_{21}(u) & L_{22}(u) - v \end{pmatrix} = 1$$

from which we conclude that

$$\left.\begin{array}{l}\alpha_1\, L_{12} = \alpha_2\, (L_{11} - v) \\ \alpha_1\, (L_{22} - v) = \alpha_2\, L_{21}\end{array}\right\} \Leftrightarrow \begin{cases} B(u) = \alpha_1^2 L_{12} - \alpha_1\alpha_2 (L_{11} - L_{22}) - \alpha_2^2 L_{21} = 0 \\ v = A(u) = L_{11} - \dfrac{\alpha_1}{\alpha_2}\, L_{12} = L_{22} - \dfrac{\alpha_2}{\alpha_1}\, L_{21}\,. \end{cases}$$

Suppose we have found a non-degenerate matrix $V(u)$ such that the Lax matrix $\widetilde{L}(u) = V(u)L(u)V^{-1}(u)$ ends up in SoV with the standard normalization vector $\boldsymbol{\alpha}_0 = (1, 0)$. That would imply separability for the matrix $L(u)$ with the normalization vector $\boldsymbol{\alpha}$ (cf equation (2.11))

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 \cdot V = (V_{11}(u), V_{12}(u))\,.$$

3. The separation

Let us recall first the construction of the 2×2 Lax matrix for the $\mathcal{D}_n$-type periodic Toda lattice with four extra parameters (Inozemtsev's case) [10, 11, 20, 14].

Given the rational classical 4×4 r-matrix of the form

$$r(u) = \frac{\kappa}{u} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

one considers two algebras: the Sklyanin quadratic algebra (S)

$$\{L^{(1)}(u), L^{(2)}(v)\} = [r(u - v), L^{(1)}(u)\, L^{(2)}(v)]$$

and the reflection equation algebra (RE)

$$\{L^{(1)}(u), L^{(2)}(v)\} = [r(u - v), L^{(1)}(u)\, L^{(2)}(v)] + L^{(1)}(u)\, r(u + v)\, L^{(2)}(v) \\ - L^{(2)}(v)\, r(u + v)\, L^{(1)}(u)\,.$$

These two algebras appeared in the quantum inverse scattering method. Their representations play an important role in the classification and studies of classical integrable systems (see, for instance, [5, 21, 23, 25] and references therein). Here the superscripted indices (1) and (2) mean standard tensoring of the 2×2 matrix $L(u)$ with the 2×2 unit matrix $\mathbb{1}$: $L^{(2)}(u) = \mathbb{1} \otimes L(u)$, $L^{(1)}(u) = L(u) \otimes \mathbb{1}$.

The following 2×2 L-operators

$$L_1(u) = \begin{pmatrix} \begin{array}{l} u^2 x_1 + u\, [i\, (x_1^2 - 1)\, p_1 + c_1 x_1 \\ \quad + c_2] + c_1 c_2 \end{array} & u\, (x_1^2 - 1) \\ \begin{array}{l} u\, (u^2 + (x_1^2 - 1)\, p_1^2 \\ \quad - 2 i p_1\, (c_1 x_1 + c_2) - c_1^2) \end{array} & \begin{array}{l} u^2 x_1 - u\, [i\, (x_1^2 - 1)\, p_1 + c_1 x_1 \\ \quad + c_2] + c_1 c_2 \end{array} \end{pmatrix}$$

and

$$L_2(u) = \begin{pmatrix} \begin{array}{l} -u^2 x_2 + u\, [i\, (x_2^2 - 1)\, p_2 + c_3 x_2 \\ \quad + c_4] - c_3 c_4 \end{array} & \begin{array}{l} u\, (u^2 + (x_2^2 - 1)\, p_2^2 \\ \quad - 2 i p_2\, (c_3 x_2 + c_4) - c_3^2) \end{array} \\ u\, (x_2^2 - 1) & \begin{array}{l} -u^2 x_2 - u\, [i\, (x_2^2 - 1)\, p_2 + c_3 x_2 \\ \quad + c_4] - c_3 c_4 \end{array} \end{pmatrix}$$

satisfy the RE algebra with $\kappa = i$. Here the (x_j, p_j) are canonical Darboux variables, i.e. the Poisson brackets are $\{p_j, x_k\} = \delta_{jk}$. These L-operators were found in [10, 11] (see also [20, 14]). They generate the $\mathcal{D}_n$-type periodic Toda lattice having four additional (singular) potential terms with the parameters c_1, c_2, c_3, c_4. Namely, consider the following Lax matrix:

$$T(u) = L_3(u) \cdot \cdots \cdot L_n(u) \cdot L_1(u) \cdot L_n^{-1}(-u) \cdot \cdots \cdot L_3^{-1}(-u) \cdot L_2(u) \tag{3.1}$$

where the L-operators $L_3, \ldots, L_n$ satisfy the (S) algebra with $\kappa = i$ and have the form:

$$L_k(u) = \begin{pmatrix} 0 & -x_k^{-1} \\ x_k & u + i p_k x_k \end{pmatrix} \qquad k = 3, \ldots, n\,.$$

The Lax matrix thus constructed describes an integrable system with the following Hamiltonian:

$$H_1 = \sum_{k=3}^{n} (x_k p_k)^2 + p_1^2\,(x_1^2 - 1) + p_2^2\,(x_2^2 - 1) - 2\sum_{k=3}^{n-1} \frac{x_k}{x_{k+1}}$$
$$+ 2\,\frac{x_2}{x_3} + 2x_1 x_n - 2ip_1\,(c_1 x_1 + c_2) - 2ip_2\,(c_3 x_2 + c_4)\,. \tag{3.2}$$

This Hamiltonian turns into that for Inozemtsev's Toda lattice (cf equation (1.2)) under the following change of variables: $x_1 = \cosh q_1,\ x_2 = \cosh q_2,\ x_j = \exp(q_j),\ j = 3, \ldots, n$, and obvious gauge-type canonical transformation for two particles (with the variables (x_1, p_1) and (x_2, p_2)) to get rid of terms linear in p_1, p_2 in (3.2).

Our problem is to separate variables in this system and restore the Lax matrix $T(u)$, equation (3.1), in terms of (new) separation variables. This is perfomed in the following three propositions.

The spectral curve has the following form:

$$\det(T(u) - v \cdot \mathbb{1}) = v^2 - v\big[(-1)^n u^{2n+2} + (-1)^n H_1\, u^{2n} + H_2\, u^{2n-2}$$
$$+ \cdots + H_n\, u^2 - 2c_1c_2c_3c_4\big] + \prod_{i=k}^{4}(u^2 - c_k^2) = 0\,. \tag{3.3}$$

Proposition 2. Let

$$V(u) = \begin{pmatrix} 1 - x_2 & u + c_3 - ip_2\,(1 - x_2) \\ 0 & \dfrac{1}{1 - x_2} \end{pmatrix}. \tag{3.4}$$

Then it is easy to verify that $V(u)$ obeys the (S) algebra with $\kappa = i$ and, moreover, it converts the matrix $L_2(u)$ into the triangular form

$$\widetilde{L}_2(u) \equiv V(-u) \cdot L_2(u) \cdot V^{-1}(u) = \begin{pmatrix} (u - c_3)(u + c_4) & 0 \\ -u\,\dfrac{1 + x_2}{1 - x_2} & (u + c_3)(u - c_4) \end{pmatrix}.$$

Proof. This is a simple and straightforward algebraic calculation. The second part of the statement is crucial for the following procedure of separation of variables and is absolutely non-trivial since we apply an *almost* similarity transformation to the boundary matrix $L_2(u)$ to put it into the triangular form (note the changed sign of the spectral parameter u). □

Proposition 3. Consider the representation of the (RE) algebra of the following form:

$$\widetilde{T}(u) = V(u) \cdot L_3(u) \cdot \cdots \cdot L_n(u) \cdot L_1(u) \cdot L_n^{-1}(-u) \cdot \cdots \cdot L_3^{-1}(-u) \cdot V^{-1}(-u)$$
$$= \begin{pmatrix} \widetilde{A}(u) & \widetilde{B}(u) \\ \widetilde{C}(u) & \widetilde{D}(u) \end{pmatrix}. \tag{3.5}$$

Then the matrix $\widehat{T}(u)$ which is similar to the $T(u)$ can be represented as follows:

$$\widehat{T}(u) \equiv V(u) \cdot T(u) \cdot V^{-1}(u) = \widetilde{T}(u) \cdot \widetilde{L}_2(u)\,. \tag{3.6}$$

Hence

$$\operatorname{tr} T(u) = (u - c_3)(u + c_4)\,\widetilde{A}(u) + (u + c_3)(u - c_4)\,\widetilde{D}(u) - u\,\frac{1 + x_2}{1 - x_2}\,\widetilde{B}(u)$$

with

$$\det \widetilde{T}(u) = (u^2 - c_1^2)(u^2 - c_2^2) \qquad \det \widetilde{L}_2(u) = (u^2 - c_3^2)(u^2 - c_4^2)$$

$$\det T(u) = \prod_{k=1}^{4}(u^2 - c_k^2)\,.$$

If we choose n zeros u_k of the polynomial $\widetilde{B}(u)$ as n separation variables:

$$\widetilde{B}(\pm u_k) = 0 \qquad \lambda_k^{\pm} = \widetilde{D}(\pm u_k) \qquad k = 1, \ldots, n \tag{3.7}$$

then they satisfy the relations

$$\{u_j, u_k\} = 0$$

$$\{u_k, \lambda_k^{\pm}\} = \pm i\,\lambda_k^{\pm}$$

$$\lambda_k^{+}\,\lambda_k^{-} = (u_k^2 - c_1^2)\,(u_k^2 - c_2^2)$$

$$\{\lambda_j^{\pm}, \lambda_k^{\pm}\} = \{\lambda_j^{\pm}, \lambda_k^{\mp}\} = \{\lambda_j^{\pm}, u_k\} = 0 \qquad j \neq k\,.$$

Moreover, from their definition it follows that they satisfy the equalities ($k = 1, \ldots, n$)

$$\operatorname{tr} T(u_k) = (u_k - c_3)(u_k + c_4)\,\lambda_k^{-} + (u_k + c_3)(u_k - c_4)\,\lambda_k^{+} \tag{3.8}$$

(the separation equations).

Proof. The matrix $\widetilde{T}(u)$ satisfies the involution

$$\widetilde{T}(-u) = \left[\det \widetilde{T}(u)\right] \cdot\, \widetilde{T}^{-1}(u) = \sigma_2\,\widetilde{T}^{t}(u)\,\sigma_2$$

or, in component-wise form,

$$\widetilde{A}(-u) = \widetilde{D}(u) \qquad \widetilde{B}(-u) = -\widetilde{B}(u) \qquad \widetilde{C}(-u) = -\widetilde{C}(u)\,.$$

Moreover, its polynomial in u entries have the degrees

$$\deg \widetilde{T}(u) = \begin{pmatrix} 2n & 2n+1 \\ 2n-1 & 2n \end{pmatrix}.$$

The matrix $\widetilde{T}(u)$ obeys the RE algebra of Poisson brackets according to the proposition 2 from [23] because $L_j(u)$, $j = 3, \ldots, n$, and $V(u)$ obey the S algebra brackets. Using our proposition 2 we establish the decomposition (3.6) for the matrix $\widehat{T}(u)$ which is similar to the Lax matrix $T(u)$. The rest of the formulae are obvious. The polynomial $\widetilde{B}(u)$ has exactly n non-trivial zeros u_k, $k = 1, \ldots, n$ (doubled by the obvious $\pm$-symmetry). The related $\lambda_k^{\pm}$ variables are defined according to (3.7). These new variables u_k, $\lambda_k^{\pm}$ satisfy the separation equations (3.8) by their definition. The calculation of all the Poisson brackets between the separation variables u_k, $\lambda_k^{\pm}$ is a standard procedure nowadays which was originally invented in [22]. Let us recall, for instance, how one calculates the brackets between u_k and λ_k^{+}. From the RE algebra for $\widetilde{T}(u)$ we have

$$-i\{\widetilde{B}(u), \widetilde{D}(v)\} = \frac{\widetilde{D}(u)\widetilde{B}(v) - \widetilde{D}(v)\widetilde{B}(u)}{u - v} + \frac{\widetilde{D}(-u)\widetilde{B}(v) + \widetilde{D}(v)\widetilde{B}(u)}{u + v}\,.$$

Combining it with the equation

$$0 = \{\widetilde{B}(u_k), \widetilde{D}(v)\} = \{\widetilde{B}(u), \widetilde{D}(v)\}\big|_{u=u_k} + \widetilde{B}'(u_k)\{u_k, \widetilde{D}(v)\}$$

we obtain

$$\{u_k, \lambda_k^+\} = \frac{-i}{\widetilde{B}'(u_k)} \left(\frac{\lambda_k^+}{u_k - v} + \frac{\lambda_k^-}{u_k + v} \right) \widetilde{B}(v)\bigg|_{v=u_k} = i\,\lambda_k^+ . \qquad \square$$

Proposition 4. The interpolation problem to restore the matrix $\widetilde{T}(u)$ in terms of new (separation) variables $u_k, \lambda_k^\pm$ has the following solution:

$$\widetilde{B}(u) = (-1)^n u \prod_{k=1}^n (u^2 - u_k^2)$$

$$\widetilde{D}(u) = (-1)^n c_1 c_2 \prod_{k=1}^n \frac{u^2 - u_k^2}{u_k^2} + \sum_{k=1}^n \left[\frac{u(u+u_k)}{2u_k^2}\,\lambda_k^+ + \frac{u(u-u_k)}{2u_k^2}\,\lambda_k^- \right] \prod_{j \neq k} \frac{u^2 - u_j^2}{u_k^2 - u_j^2}$$

$$\widetilde{A}(u) = \widetilde{D}(-u) \qquad \widetilde{C}(u) = \frac{\widetilde{A}(u)\widetilde{D}(u) - (u^2 - c_1^2)(u^2 - c_2^2)}{\widetilde{B}(u)} .$$

Proof. The formula for $\widetilde{B}(u)$ is obvious. The polynomial $\widetilde{D}(u)$ of degree $2n$ is restored in terms of the separation variables by interpolation with $2n+1$ data of the form

$$\widetilde{D}(\pm u_k) = \lambda_k^\pm \qquad \widetilde{D}(0) = c_1 c_2 . \qquad \square$$

Now we can derive, in principle, the formulae connecting old and new variables. For instance, noting that $\widetilde{D}(u)$ has the asymptotics

$$\widetilde{D}(u) = \frac{(-1)^n}{1 - x_2}\, u^{2n} + \cdots \qquad u \to \infty$$

we find that

$$\frac{1}{1 - x_2} = \frac{c_1 c_2}{\prod_{k=1}^n u_k^2} + (-1)^n \sum_{k=1}^n \frac{\lambda_k^+ + \lambda_k^-}{2u_k^2 \prod_{j \neq k}(u_k^2 - u_j^2)} .$$

We can express some other combinations of the initial variables in terms of new (separation) variables, comparing the coefficients of the entries of $\widetilde{T}(u)$ in both representations. Considering the $\operatorname{tr} T(u)$, we could also obtain the expressions for the integrals of motion $H_1, \ldots, H_n$ in terms of the separation variables.

Corollary 1. The separating normalization vector for the $\mathcal{D}_n$-type periodic Toda lattice with the Hamiltonian (3.2) and with the Lax matrix (3.1) has the form

$$\boldsymbol{\alpha} = (1 - x_2,\; u + c_3 - i p_2 (1 - x_2)) .$$

The separation variables u_k and $v_k^\pm \equiv (u_k \pm c_3)(u_k \mp c_4)\,\lambda_k^\pm$, $k = 1, \ldots, n$, are sitting on the spectral curve (3.3) of the Lax matrix $T(u)$, equation (3.1):

$$(v_k^\pm)^2 - v_k^\pm \operatorname{tr} T(u_k) + \det T(u_k) = 0$$

i.e.

$$v_k^+ + v_k^- = \operatorname{tr} T(u_k) \qquad v_k^+ v_k^- = \det T(u_k) .$$

They have the following Poisson brackets

$$\{u_k, v_k^\pm\} = \pm i\, v_k^\pm .$$

Remark 1. The (obvious) alternative choice of the separating normalization vector follows if we convert the matrix $L_1(u)$ instead of the matrix $L_2(u)$ (cf proposition 2) to triangular form. This would correspond to interchanging two edge particles in the lattice.

It would be interesting to construct (explicitly) the generating function $F(u|x)$ of this separating canonical transform.

If we introduce the canonically conjugate variables π_j

$$\{\pi_j, u_k\} = \delta_{jk}$$

then we can put

$$v_k^{\pm} = [\det T(u_k)]^{\frac{1}{2}} \exp(\mp i\, \pi_k)$$

and obtain the separation equations in the form

$$2\,[\det T(u_k)]^{\frac{1}{2}} \cos(\pi_k) = \operatorname{tr} T(u_k)\,.$$

Hence, the action variables $S_k(H_1, \ldots, H_n)$ have the form

$$S_k(H_1, \ldots, H_n) = \oint_{\alpha_k} \arccos\left(\frac{\operatorname{tr} T(u)}{2[\det T(u)]^{\frac{1}{2}}}\right) \mathrm{d}u \qquad k = 1, \ldots, n$$

where α_k are the α-cycles on the Riemannian surface of $\sqrt{\operatorname{tr}^2 T(u) - 4 \det T(u)}$.

One can obtain the quasiclassical spectrum $H_k(N_1, \ldots, N_n)$ of the integrals of motion $H_1, \ldots, H_n$ (cf [9]) by inverting the integrals (Bohr–Sommerfeld quantization)

$$S_k(H_1, \ldots, H_n) = h\, N_k \qquad k = 1, \ldots, n$$

where the N_k are the quantum numbers $N_k = 1, 2, 3, \ldots$. Obtaining the true discrete spectrum of the integrals of the quantum $\mathcal{D}_n$-type periodic Toda lattice is the problem of quantum separation of variables.

4. Concluding remarks

We refer the reader to the review [25] (cf also [15]) where it was demonstrated that the simplest choice of the normalization vector $\boldsymbol{\alpha}$, when one of the components of the Baker–Akhiezer function $f(u)$ (for instance the first one) is equal to 1, i.e. when

$$\boldsymbol{\alpha} = (1, 0, \ldots, 0) \tag{4.1}$$

provides a SoV for many integrable systems of $\mathcal{A}_n$ type. If a chosen integrable system cannot be separated with this simplest normalization, and this usually means that its Lax matrix has some extra symmetries/involutions (i.e. is of the $\mathcal{BC}_n$ or $\mathcal{D}_n$ type or obeys an elliptic r-matrix), then the main problem is to find the proper $\boldsymbol{\alpha}$. For the time being there is no theory to give a general prescription for finding the correct normalization vector $\boldsymbol{\alpha}$ in those cases. However, one practical rule can be suggested. Usually, if one looks at the poles of the Baker–Akhiezer function with the simplest normalization (4.1), one finds that there are too many poles and they do not respect the symmetry presenting in the problem. Then the rule is the following: take an ansatz for $\boldsymbol{\alpha}(u)$ with some dependence on u and with some indeterminates in it, derive equations for those indeterminates demanding that (a) $f(u)$ with such a normalization has the correct number of moving poles respecting involutions of the spectral curve and (b) all extra poles are equal to constants. Then solve the equations . . . .

In this paper we applied this approach to the $\mathcal{D}_n$-type periodic Toda lattice with four additional singular terms in the potential. This system cannot be separated with the simplest choice of the normalization vector $\boldsymbol{\alpha}$ (4.1), so we have derived the correct normalization $\boldsymbol{\alpha}$ producing the SoV. For some of the root systems the separating normalization vector is a constant vector (cf the $\mathcal{BC}_n$ case in [11, 20, 14]). For the generic $\mathcal{D}_n$ case the separating $\boldsymbol{\alpha}(u)$ depends on the spectral parameter u and on the phase variables, so it is dynamical. We think that it is an important feature of this kind of problems (those with extra involutions), that the separating choice of $\boldsymbol{\alpha}$ is not completely arbitrary, as it was for some of the $\mathcal{A}_n$-type systems, but is quite unique and dynamical.

The specific situation with the $\mathcal{D}_n$-type periodic Toda lattice, i.e. that the correct $\boldsymbol{\alpha}$ is u-dependent and dynamical, is surely connected with the fact that we use dynamical boundary $L_{1,2}$-matrices in constructing the corresponding Lax matrix $T(u)$ (3.1) for this case.

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